RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 48, NUMBER 2 AUGUST 1993

Scale invariance of nonconserved quantities in driven systems

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(Received 5 May 1993)

Noisy nonequilibrium systems involving locally conserved quantities typically exhibit generic scale invariance—infinite correlation lengths and the associated algebraic decay of correlations without the tuning of external parameters. It is shown here that if such a conserved field, ϕ_c, is coupled linearly to a nonconserved one, ϕ_n, generic power-law decays are induced in the correlations of ϕ_n. When symmetry prevents linear coupling, correlations of the ϕ_n field decay exponentially under generic conditions, unless ϕ_n experiences a broken symmetry, in which case linear coupling and hence algebraic decays can be generated. Numerical support for these results in simple conserving coupled map lattices is presented.

PACS number(s): 64.60.Ht, 05.40.+j, 05.60.+w, 46.10.+z

I. INTRODUCTION

In equilibrium systems at finite temperature, scale invariance—infinite correlation lengths and the resulting power-law decay of spatial and temporal correlations—typically occur only at critical points or in systems with certain continuous symmetries (such as the symmetry of an interface under uniform translations). In certain stochastic nonequilibrium systems, however, scale invariance can occur generically, i.e., without the tuning of a parameter [1–4]. Fluids driven by a temperature gradient [5] were the first examples of this phenomenon to be discovered, though more recently driven diffusive systems [6] and driven interfaces such as stochastic model sandpiles [1,7] have also been shown to exhibit generic scale invariance.

For stochastic nonequilibrium systems with a single-component field in $d>1$ dimensions, local conservation of that field has been shown [1–3] to be a necessary and almost sufficient condition for algebraic behavior of correlations to occur generically. In this paper we consider classical nonequilibrium systems with two or more coupled fields, some of which are conserved and others not. Such problems (e.g., a nonconserved magnetization coupled to a conserved energy) abound. They have been treated in equilibrium systems near critical points, where the conservation law can significantly affect the dynamics [8], but not studied systematically in nonequilibrium situations. We show here that a *linear* coupling between the conserved and nonconserved fields is sufficient to induce algebraic decays of correlations in the nonconserved variable under generic conditions. The exponents governing the decay of correlations of the conserved and nonconserved fields are, moreover, identical in all the instances we have studied. In cases where symmetry prohibits linear coupling, nonlinear coupling is typically *insufficient* to induce algebraic behavior. In such situations, however, spontaneous breaking of symmetry (e.g., the development of magnetic order in the example above) often produces a linear coupling which then induces generic scale invariance in the nonconserved quantity. We believe that this phenomenon occurs commonly in the routes to chaos of conserving systems, in a manner illustrated below.

II. LANGEVIN EQUATIONS AND ANALYSIS

These results apply to systems whose long-wavelength behavior is accurately captured by Langevin equations for the evolution of course-grained fields with Gaussian noise terms. Such equations are believed to be appropriate, at sufficiently long length and time scales, for describing most systems with microscopic interactions of finite range. To derive our results, we start with the simplest linear model of fluctuating hydrodynamics involving the coupling of two fields, $\phi_c(\mathbf{x},t)$ and $\phi_n(\mathbf{x},t)$, which are respectively conserved and not conserved:

$$\frac{\partial\phi_c}{\partial t}=Q_1\phi_c+Q_2\phi_n+\eta_c \ , \tag{1}$$

$$\frac{\partial\phi_n}{\partial t}=-r_n\phi_n-r_c\phi_c+Q_3\phi_n+Q_4\phi_c+\eta_n \ . \tag{2}$$

Here Q_i for $i=1,2,3,4$ are differential operators taken for now to satisfy reflection invariance, $\mathbf{x}\to-\mathbf{x}$, and whatever other symmetries the problem in question dictates. For example, for problems respecting the symmetry of a hypercubic lattice in d space dimensions, Q_i would take the form $a_i\nabla^2+b_i\sum_{j=1}^{d}\nabla_j^4+e_i(\nabla^2)^2+\cdots$, for constants a_i, b_i, and e_i. The quantities $r_n(>0)$ and r_c are constants, while $\eta_c(\mathbf{x},t)$ and $\eta_n(\mathbf{x},t)$ are independent random noise variables with zero mean, whose correlations are taken to be Gaussian. As discussed in the context of Langevin equations with a single conserved field [1,2], η_c can be chosen either to conserve the field ϕ_c, as would be appropriate if ϕ_c represented a conserved density (as in driven diffusive systems [6]), or not, as in certain driven interfaces (such as stochastic sandpile systems [1,7]), for which Eq. (1) must be invariant under the transformation $\phi_c \to \phi_c + \text{const}$. Under generic conditions the noise η_n is nonconserving [9].

The straightforward solution of Eqs. (1) and (2) shows that for the asymptotically large distances and times in which we are interested, the dominant terms in the equations for the Fourier transformed field variables $\phi_c(\mathbf{k},\omega)$ and $\phi_n(\mathbf{k},\omega)$ yield simply

$$\phi_c(\mathbf{k},\omega) \sim \eta_c/[-i\omega - Q_n(\mathbf{k})] , \quad (3)$$

$$\phi_n(\mathbf{k},\omega) \sim -r\phi_c(\mathbf{k},\omega) , \quad (4)$$

where $r \equiv r_c/r_n$ and $Q_n(\mathbf{k}) \equiv Q_1(\mathbf{k}) - rQ_2(\mathbf{k}) \sim -a_n k^2 + b_n \sum_{j=1}^{d} k_j^4 + \cdots$. Here $a_n \equiv a_1 - ra_2$ and $b_n \equiv b_1 - rb_2$, and we consider only the stable situation where $a_n > 0$ and $b_n < 0$. Thus, aside from the values of the coefficients a_n and b_n, the small-k and -ω behavior of both ϕ_c and ϕ_n is the same as one would obtain for the conserved field $\phi_c(\mathbf{k},\omega)$ in the absence of any coupling to ϕ_n, i.e., if Q_2 were equal to 0. Hence a necessary and sufficient condition for $\phi_n(\mathbf{x},t)$ to exhibit generic algebraic decays of correlations in model (1),(2) is that $\phi_c(\mathbf{x},t)$ do so when $Q_2 = 0$. Thus the problem has been reduced to that of the single conserving field considered in Ref. [2], where it was shown that for lattice systems with $d > 1$, power-law correlations obtain under generic conditions [9]. Obviously, ϕ_n is just "slave" [10] to ϕ_c [11], so correlations of the two fields decay with identical exponents, as do equal-time cross correlations of ϕ_c and ϕ_n.

To study the effect on these results of including nonlinearities, we restrict ourselves to the more common situation where η_c represents conserving noise. Imagine adding to Eqs. (1) and (2) all analytic nonlinear terms consistent with the symmetry of the problem. Through renormalization-group (RG) calculation [8,12], these can all be shown to be irrelevant (i.e., not to alter the asymptotic behavior of correlations), at least for small values of the coefficients of the nonlinearities, where perturbative RG analysis is valid. To understand this, recall [12] that the most relevant nonlinear terms are always those with the fewest powers of the field and the fewest derivatives, viz., terms of the form $\nabla^2(\phi_c^2)$, $\nabla^2(\phi_n^2)$, and $\nabla^2(\phi_c\phi_n)$ in Eq. (1), and ϕ_c^2, ϕ_n^2, and $\phi_c\phi_n$ in Eq. (2). With the standard RG scaling, $x = bx'$, $t = b^z t'$, and $\phi_c = b^\zeta \phi_c'$, where $b\,(>1)$ is the length rescaling parameter, and z and ζ are the dynamical [8] and field rescaling exponents, respectively, one readily shows that $z = 2$, and that the coefficient, λ_{cc}, say, of the $\nabla^2(\phi_c^2)$ term in (1) scales like $\lambda_{cc}' = b^{-d/2}\lambda_{cc}$, i.e., is irrelevant for all d. To preserve the stable solution of the linear problem, the rescaling of ϕ_n must be the same as that of ϕ_c; i.e., $\phi_n = b^\zeta \phi_n'$. It follows straightforwardly from this unusual scaling that the $\partial\phi_n/\partial t$ and η_n terms in Eq. (2) are both irrelevant, as are all other nonlinear terms in Eqs. (1) and (2). For example, the coefficients (λ_n, say) of the three ϕ^2 terms of (2) all scale like $\lambda_n' = b^{(z-d-2)/2}\lambda_n = b^{-d/2}\lambda_n$. One concludes that the results of the linear theory continue to hold in the presence of nonlinearities.

One can of course treat in similar fashion the less symmetric situations that arise commonly in nonequilibrium systems. Consider, e.g., driven diffusive systems [6] or stochastic sandpiles [1,7] where particles are both conserved and driven in a particular (say, the x) direction, so that mirror symmetry in that direction is violated. If the conserved field ϕ_c is imagined coupled to a nonconserved (e.g., magnetization) field ϕ_n, the resulting linear equations of motion continue to take the form of (1) and (2); however, the differential operators Q_i now reflect the lack of $\mathbf{x} \to -\mathbf{x}$ symmetry: $Q_i \sim c_i \nabla_x + a_i \nabla^2 + \cdots$ for constants c_i and a_i. While the leading generic nonlinear terms in the ϕ_n equation remain ϕ_n^2, $\phi_n\phi_c$, and ϕ_c^2, the leading nonlinearities in the ϕ_c equation now also incorporate the absence of reflection symmetry; they are $\nabla_x(\phi_c^2)$, $\nabla_x(\phi_n^2)$, and $\nabla_x(\phi_c\phi_n)$.

To analyze these equations, it is helpful to eliminate the ∇_x terms from the linear pieces through the following transformation to new fields $\tilde{\phi}_c$ and $\tilde{\phi}_n$ (dependence on the spatial coordinates transverse to x being suppressed):

$$\tilde{\phi}_c(x,t) = \phi_c(\tilde{x},t) + \tilde{c}\nabla_x \phi_n(\tilde{x},t) , \quad (5)$$

$$\tilde{\phi}_n(x,t) = \phi_n(\tilde{x},t) , \quad (6)$$

where $c \equiv c_1 - rc_2$, $\tilde{c} \equiv c_2/r_n$, and $\tilde{x} \equiv x - ct$. This is easily seen to remove the ∇_x terms (i.e., those proportional to c_1 and c_2) from the equation for $\tilde{\phi}_c$, while altering only the coefficients but not the form of either the remaining pieces of the Q_i or the nonlinearities. Aside from the ∇_x terms in the $\tilde{\phi}_n$ equation, which play no role at long distances and times, the linear equations thus become identical to (1) and (2), and hence show the same induced scale invariance in $\tilde{\phi}_n$. (Note that the transformation can in principle produce a negative coefficient of the $\nabla_x^2\tilde{\phi}_n$ term in the $\tilde{\phi}_n$ equation. We consider only parameter values for which this coefficient is positive, thus ensuring the stability of the equations.)

The nonlinearities are again treated with RG methods. Once more scaling $\tilde{\phi}_n$ the same way as $\tilde{\phi}_c$ and restricting oneself to noise η_c that conserves, one finds that all terms in the $\tilde{\phi}_n$ equation except those simply proportional to $\tilde{\phi}_c$ and $\tilde{\phi}_n$ are irrelevant. Again, it follows immediately that the asymptotic equal time correlations of both the ϕ_c and ϕ_n fields are the same as would be obtained for ϕ_c in the absence of any coupling to ϕ_n. Thus the asymptotic power-law correlations induced in ϕ_n are simply those of the conserved field ϕ_c. Note, however, that, unlike in the reflection symmetric case where the nonlinear terms played no role, here the $\nabla_x\phi_c^2$ nonlinearity is well known [1,2,13] to become marginally irrelevant at $d = 2$, producing logarithmic corrections to the algebraic decays in $d = 2$, and nontrivial changes in the exponents for $1 < d < 2$. (For $d = 1$ the conservation law typically does not produce algebraic correlations even in ϕ_c when the noise η_c is conserving [2].)

Next consider situations where symmetry prohibits linear coupling between the conserved and nonconserved fields. An elementary example is provided by nonequilibrium models with coupled fields representing, e.g., a conserved density, ϕ_c, and a nonconserved ferromagnetic Ising order parameter, ϕ_n. If the problem has reflection and up-down ($\phi_n \to -\phi_n$) symmetry, then the equations of motion assume the form

$$\frac{\partial \phi_c}{\partial t} = \nabla^2(\phi_c + \lambda_{nn}\phi_n^2 + \lambda_{cc}\phi_c^2) + \cdots + \eta_c \ , \quad (7)$$

$$\frac{\partial \phi_n}{\partial t} = -r_n\phi_n + \gamma\nabla^2\phi_n + \lambda_{nc}\phi_n\phi_c + \lambda_{nnn}\phi_n^3 + \cdots + \eta_n \ . \quad (8)$$

Note that owing to the up-down symmetry, these equations decouple at linear order, unlike Eqs. (1) and (2). Hence, to this order ϕ_c exhibits generic power laws [provided the lattice anisotropy is manifest either in the ellipsis in (7) or the correlations [9] of η_c], but cannot induce scale invariance in ϕ_n. Symmetry-allowed nonlinearities, the leading terms of which are shown in Eqs. (7) and (8), likewise do not induce algebraic behavior in the ϕ_n correlations. To see this, note that in the linear theory correlations of ϕ_n decay exponentially with correlation length $(\gamma/r_n)^{1/2}$. Under standard dynamical RG transformation, the "mass" r_n increases, approaching the fixed point at $r_n=\infty$, at which correlations of ϕ_n remain exponential. It follows that the field ϕ_n does not become critical; one can integrate it out of Eqs. (7) and (8), producing a single equation for the conserved field ϕ_c; the algebraic decays of correlations of ϕ_c are then guaranteed by the arguments of Ref. [2] to survive the inclusion of nonlinearities.

The situation becomes more interesting in the ordered phase of Eqs. (7) and (8) (i.e., $r_n<0$, roughly), where ϕ_n spontaneously develops a nonzero expectation value $M \equiv \langle \phi_n \rangle$. Writing $\phi_n = M + \hat{\phi}_n$, one finds that Eqs. (7) and (8) written in terms of $\hat{\phi}_n$ take the form of Eqs. (1) and (2); i.e., ϕ_c and $\hat{\phi}_n$ become *linearly* coupled, the coupling constants being proportional to M. In this case, therefore, power laws are induced in the correlations of $\hat{\phi}_n$, as described above.

III. COUPLED-LATTICE-MAP REALIZATION

This phenomenon of power laws induced by the combination of conservation and broken symmetry is likely to be a common one in driven systems. The point is that when systems are prevented from responding to external driving in a particular mode ($q=0$, say), because of conservation, they frequently respond by breaking symmetry in a different, nonconserved mode [14]. Let us illustrate this point with a specific example—a simple system of coupled lattice maps with conservation. The model is defined by the simultaneous updating of the equation

$$S_{t+1}(i) = S_t(i) + \frac{\nu}{4}\sum_j [f(S_t(j) - S_t(i))] + \frac{\alpha}{2}[S_t^2(i) - S_t^2(i+2\hat{\mathbf{x}})] + \eta_t(i) \ . \quad (9)$$

Here $S_t(i)$ is the dynamical variable on site i of a two-dimensional square lattice at time t, j denotes the four nearest neighbors of i, $\hat{\mathbf{x}}$ is a unit vector along the x axis, and $f(S)=S-S^3$; ν (which regulates the nonlinear diffusive coupling between sites) and α are constants; $\eta_t(i)$ is a random noise variable generated from a second set of noise variables $\hat{\eta}_t(i)$, so as to preserve the local conservation of the $S_t(i)$ inherent in the deterministic part of (9) [i.e., $\rho \equiv \sum_i S_t(i)/N$ is independent of t, where N is the number of sites]. Specifically, $\eta_t(i) = \hat{\eta}_t(i+\hat{\mathbf{x}}) - \hat{\eta}_t(i-\hat{\mathbf{x}})$; the $\hat{\eta}_t(i)$ are chosen independently and randomly from either a Gaussian distribution of width σ, or a distribution uniform for $|\hat{\eta}| \le \sigma$ and zero otherwise.

This model, introduced in Ref. [15], has a complex phase diagram with several different phases. The phase of interest here is a spatially ordered, checkerboard ("staggered"), temporal two-cycle, in which the odd and even sublattices interchange values at each time step. This phase clearly breaks spontaneously the (discrete) spatial and temporal translational invariance of the rule (9). It is convenient to eliminate the complication of the temporal two-cycle oscillations by iterating (9) once to express $S_{t+2}(i)$ as a function of the $\{S_t(j)\}$. In terms of this once-iterated rule, the phase in question is a steady-state checkerboard wherein the spatial broken symmetry is characterized by a nonzero expectation value of the staggered order parameter, the $\mathbf{q}=(\pi,\pi)$ Fourier component of S: $M \equiv \langle S_t(\mathbf{q}=(\pi,\pi))\rangle$. Thus the once-iterated model (9) is naturally described by two coupled fields, one conserved field [analogous to ϕ_c in Eq. (7)], representing the fluctuations near $\mathbf{q}=(0,0)$, and the other [analogous to ϕ_n in (8)], representing the nonconserved staggered fluctuations around $\mathbf{q}=(\pi,\pi)$. Since model (9) has the $\phi_n \to -\phi_n$ symmetry of Eqs. (7) and (8), these equations provide a correct long-wavelength description of the once-iterated version of (9).

The broken-symmetry checkerboard phase of (9) is therefore equivalent at long wavelengths to the ordered phase of (7) and (8), in which $M \equiv \langle \phi_n \rangle \neq 0$. We argued above that this symmetry breaking induces a linear coupling between $\hat{\phi}_n$ and ϕ_c, i.e., between the fluctuations near $\mathbf{q}=(0,0)$ and $\mathbf{q}=(\pi,\pi)$, and hence produces power-law correlations in $\hat{\phi}_n$ as well as ϕ_c. Given the form of the noise in (7) and (8), it is straightforward to show [2,15] at the linear approximation level that both $G_c(\mathbf{r}) \equiv \langle \phi_c(\mathbf{r})\phi_c(\mathbf{0})\rangle$ and $G_n(\mathbf{r}) \equiv \langle \hat{\phi}_n(\mathbf{r})\hat{\phi}_n(\mathbf{0})\rangle$ decay like $1/r^2$ for large r, correlations in the x and y directions having opposite signs [2,3]. It follows that the equal-time correlations, $G(i) \equiv \langle S_t(i)S_t(0)\rangle$, of the field $S_i(t)$ of model (9) decay algebraically with a mixed uniform and staggered character, i.e.,

$$G(x) \sim [A + B(-1)^x]/|x|^2 \quad (10)$$

for sites whose separation is in the x direction, say, where

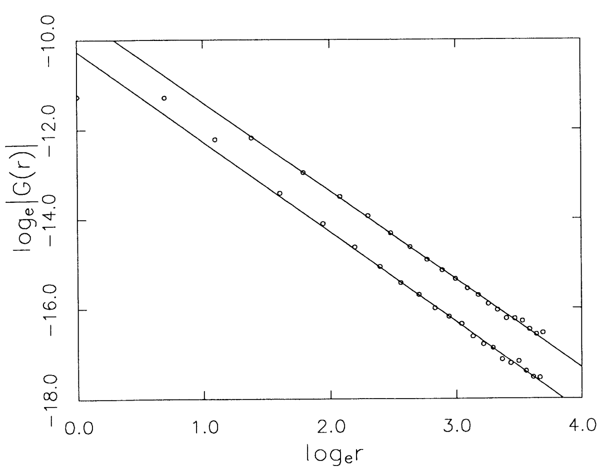

FIG. 1. Log-log plot of $|G(r\hat{\mathbf{x}})|$ vs r in the two-cycle checkerboard phase of model (9), with $\nu=1.30$, $\alpha=0.25$, $\rho=0.10$, and $\sigma=0.02$, on an 80×80 lattice with periodic boundary conditions. The straight lines, drawn as guides to the eye, have slope -2. The staggering of points for even and odd r shows "induced scale invariance" in the $\mathbf{q}=(\pi,\pi)$ mode.

A and B are constants.

To test these conclusions we have simulated [15] model (9) numerically, computing the correlation function $G(x)$. The results, shown in Fig. 1, show the staggered behavior predicted in Eq. (10). Interestingly, such behavior is observed both in the presence of external noise [$\sigma\neq0$ in (9)], or in the absence of noise, in a two-cycle checkerboard phase with chaotic fluctuations [15]. (It is data for the noisy case that are shown in the figure.) How generally the induced scale invariance discussed here holds in the presence of deterministic chaotic fluctuations rather than external noise is an intriguing open question.

Even with the two coupled fields we have considered so far, there are obviously different possible symmetries that might occur in different physical situations. In systems with three or more coupled fields, at least one of which is conserved, the number of possibilities grows rapidly. The spirit of this paper being illustrative rather than exhaustive, we have made no attempt to classify all of these. We point out only that the simplest generalization of Eqs. (1) to three linearly coupled fields, one of which, ϕ_c, is conserved while the others are not, produces generic powers in the correlations of the nonconserved fields, as in the two-field case. It therefore seems clear that induced scale invariance through either direct or broken-symmetry-generated linear coupling is quite general. Application of the methods discussed here should enable one quickly to decide whether any given nonlinear problem with arbitrary numbers of fields and symmetry exhibits this phenomenon.

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